Taylor & Francis
Taylor & Francis Group

International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

Frequency-limited H∞-controller order reduction for linear parameter-varying systems

Hossni Zebiri, Benjamin Mourllion & Michel Basset

To cite this article: Hossni Zebiri, Benjamin Mourllion & Michel Basset (2017) Frequency-limited H∞-controller order reduction for linear parameter-varying systems, International Journal of Control, 90:9, 2031-2046, DOI: 10.1080/00207179.2016.1233482

To link to this article: https://doi.org/10.1080/00207179.2016.1233482

View supplementary material

Accepted author version posted online: 13 Sep 2016.
Published online: 27 Sep 2016.

Submit your article to this journal

Article views: 124

View related articles

View Crossmark data

Citing articles: 2 View citing articles

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=tcon20

INTERNATIONAL JOURNAL OF CONTROL, 2017
VOL. 90, NO. 9, 2031–2046
http://dx.doi.org/10.1080/00207179.2016.1233482

Taylor & Francis
Taylor & Francis Group

Frequency-limited $\mathscr{H}_\infty$-controller order reduction for linear parameter-varying systems

Hossni Zebiri, Benjamin Mourllion and Michel Basset

Laboratoire Modélisation, Intelligence, Processus et Systèmes MIPS EA2332, Université de Haute-Alsace, Mulhouse, France

ABSTRACT
In this paper, a controller order reduction method for linear parameter-varying systems is presented. The proposed method is based on the frequency-weighted balanced truncation technique, which has the advantage to reduce the order in a specific frequency range. The approach is discussed and is proved to preserve the closed-loop stability with a guaranteed upper error bound. Effectiveness and performance of the obtained reduced-order controller are investigated by applying it to an automotive semi-active suspension control. The obtained simulation results show that objectives such as the road handling and the passenger comfort realised with the reduced-order controller are kept in the same performance level as with the full-order controller. Moreover, a comparison with an other-order reduction method is shown and confirms the advantage of the developed method.

ARTICLE HISTORY
Received 19 February 2016
Accepted 2 September 2016

KEYWORDS
Linear parameter-varying system; $\mathscr{H}_\infty$-control; generalised reachability and observability Gramians; balanced truncation; semi-active suspension control

Notations

$\mathbb{R}$	fields of real numbers
$\prec(\preceq), \succ(\succeq)$	negative (semi-negative), positive (semi-positive) definite
$\mathbb{S}^n_{\succ 0}$	cone of $n \times n$ symmetric positive definite matrices
$\mathcal{R}, \mathcal{O}$	reachability and observability Gramians
$P(\rho), Q(\rho)$	generalised reachability and observability Gramians
$\mathbf{P}_\Omega(\rho), \mathbf{Q}_\Omega(\rho)$	frequency-limited generalised reachability and observability Gramians
σ_i	ith generalised Hankel singular value
γ_i	ith frequency-limited generalised Hankel singular value
S^*	complex conjugate transpose of S
A^{-T}	$(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$
I_n	$n \times n$ identity matrix

1. Introduction

Most of the physical systems are inherently nonlinear and almost all of them have parameter-dependent representations. Considering those aspects and the advantages of linear control theory, more and more control strategies use linear parameter-varying (LPV) models. In the same way, and as operating conditions may change, the closed-loop performance designed by an linear time-invariant (LTI) controller may be degraded. To overcome this problem, the design of parameter-dependent controller is more suitable. Consequently, a major part of the recently developed control strategies is based on optimal and/or robust control. The $\mathscr{H}_\infty$-control strategies have the advantage to design controllers achieving stabilisation with guaranteed performance. However, these techniques usually produce high-order controllers. The design and the use of such high-order controllers can lead to numerical difficulties. This is why their use is still limited in the engineering field. Thus, for these practical reasons, low-order controllers are particularly preferred: their simple dynamics are easier to manage, they require less computing cost and simpler software can be finally implemented. Then, an order reduction step appears of great interest. Simultaneously, the reduction process should always preserve the closed-loop stability and should guarantee a level of performance close to the one obtained with the full-order controller. In a general way, a low-order controller can be obtained through the direct method or the indirect ones (see Figure 1). Considering the direct method, the final low-order controller is obtained directly from the high-order plant and the order is generally fixed beforehand (Ankelhed, Helmersson, & Hansson, 2011; Apkarian & Gahinet, 1995).

In the indirect ways, the low-order controller is the result of a model or a controller order reduction step: either a reduced-order plant is found and then

CONTACT Hossni Zebiri ✉ hossni.zebiri@uha.fr

© 2017 Informa UK Limited, trading as Taylor & Francis Group

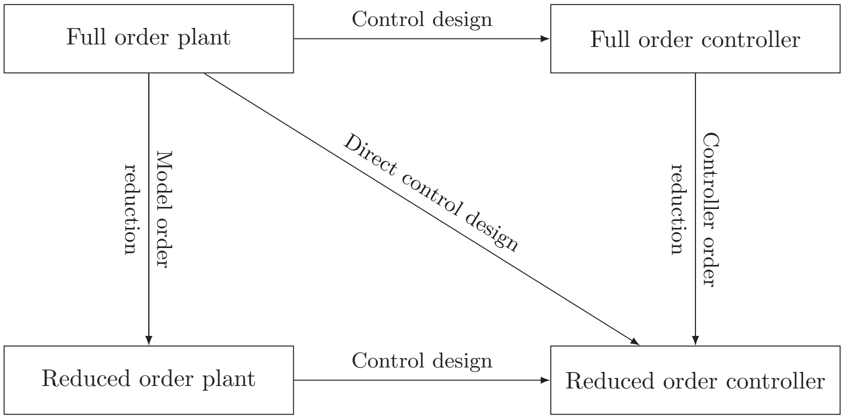

Figure 1. Controller order reduction scheme (Anderson & Liu, 1989).

a controller is designed or a high-order controller is first designed and then a controller order reduction step is performed. The last method is widely used to obtain reduced-order controllers (Goddard, 1995; Kong, 2012). For LTI systems, a large investigation has been made on model order reduction procedures. In this domain, methods based on balanced realisation are extensively used (Gugercin & Antoulas, 2004). First introduced by Mullis and Roberts (1976) and later in the systems and control literature by Moore (1981), the balanced truncation (BT) has been a significant contribution to system theory. In Pernebo and Silverman (1982), the stability of the reduced-order model is preserved. Then, Glover (1984) has proposed an $\mathscr{H}_\infty$-norm upper bound of the approximation error.

Based on this, the BT method has been proposed to reduce the order of controllers for LTI systems (Liu & Anderson, 1989; Zhou, D'Souza, & Cloutier, 1995). The extension of these reduction techniques to LPV systems is still in progress. Generally, these methods substitute the use of LTI Gramians by using parameter-/time-varying equivalents (Sandberg & Rantzer, 2004; Wood, Goddard, & Glover, 1996). A generalised method with unbounded rate parameter model is given in El-Zobaidi and Jaimoukha (1998). In the same way, an effective BT method for $\mathscr{H}_\infty$-LPV controller order reduction is proposed in Widowati, Riyanto, and Saragih (2004).

The BT produces good approximations in high frequency, whereas for control propose, an acceptable reduction is often required at intermediate and/or low frequencies. To solve this problem, the so-called frequency-weighted balanced truncation (FWBT) has been proposed first for LTI systems in Enns (1984). In this approach, two weighting functions are included in the procedure to reduce the model in a certain frequency range. However, the stability preservation is not guaranteed anymore. To tackle this problem, a modified version assuming that the full-order model does not have any common pole/zero with the weighting functions is proposed in Lin and Chiu (1990). Then, Wang, Sreeram, and Liu (1999) and Varga and Anderson (2001) have developed a new approach based on the Enns method without the previous assumption to guarantee the stability. It gives also an upper error bound. Other upper bounds have also been established by Sreeram and Anderson (1995), Kim, Anderson, and Madievski (1995) and Zhou (1995). The frequency range is explicitly defined instead of the two frequency weights in Gawronski and Juang (1990). More recently, in Gugercin and Antoulas (2004), the method is modified and an error upper bound to the relative error is given. Note that the stated techniques are developed for the stable LTI systems and no guarantee is given to preserve the passivity of the systems. Recent work in this direction are given in Li, Yin, and Gao (2014) and Li, Yu, and Gao (2015), where the $\mathscr{H}_\infty$-norm of the error is bounded and the positive realness of the obtained model is guaranteed.

Based on these recent results in the literature, the aim of this paper is to adapt the FWBT method (Gugercin & Antoulas, 2004) in order to reduce the order of an $\mathscr{H}_\infty$-LPV controller. For the development of this approach, the generalised Gramian framework is used (Shaker and Wisniewski, 2011).

The paper is organised as follows: the LPV systems, their properties and the the full-order $\mathscr{H}_\infty$-LPV controller design are introduced in Section 2. In Section 3. the generalised Gramian framework for LPV systems is given and the main proposed method is detailed and discussed. To emphasise the effectiveness of the developed method, the obtained reduced-order controller is evaluated on a practical engineering problem: the control of an automotive semi-active suspension. The performances of the obtained reduced-order controller are investigated and compared with another reduced-order method based on BT approach (Widowati, Riyanto, & Bambang, 2006) in Section 4. A final conclusion is given in Section 5.

2. Problem statement

2.1 LPV systems

LPV models represent systems whose state-space descriptions are known functions of varying parameters. This notion of LPV system has been introduced in the context of gain scheduling (Shamma & Athans, 1990) and has been also widely used in control design for nonlinear systems (Packard & Kantner, 1996).

Considering a compact subset $\mathbf{\Delta}_\rho \subset \mathbb{R}^s$, an LPV system can be described by the following state-space realisation:

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) = C(\rho(t))x(t) + D(\rho(t))u(t) \end{cases} \quad \forall t \geq 0 \tag{1}$$

where $A : \mathbf{\Delta}_\rho \to \mathbb{R}^{n\times n}, B : \mathbf{\Delta}_\rho \to \mathbb{R}^{n\times m}, C : \mathbf{\Delta}_\rho \to \mathbb{R}^{p\times n}$ and $D : \mathbf{\Delta}_\rho \to \mathbb{R}^{p\times m}$. The state-space representation (1) can be also written as

$$G(\rho) = \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right] \tag{2}$$

The exogenous parameter vector ρ is varying in a bounded set $\mathcal{P}_\rho$ defined as

$$\mathcal{P}_\rho = \{\rho : \begin{array}{l} \mathbb{R}^+ \to \mathbf{\Delta}_\rho \\ t \quad \mapsto \rho(t) \end{array} \mid \underline{\rho}_i(t) \leq \rho_i(t) \leq \overline{\rho}_i(t), i = 1, 2, \ldots, s\} \tag{3}$$

where ρ_i is the ith component of the ρ vector.

The ρ function is not known in advance but is assumed to be bounded and measurable online. This assumption ensures that all trajectories of the system are contained in those ones of the LPV model.

It is noted that some LPV models could be written as the function of system states (Sename, Gaspar, & Bokor, 2013). Such models known as *quasi-LPV* systems are used for the modelling of nonlinear systems.

Definition 2.1 (Quadratic stability)**:** System (1) is said to be quadratically stable if the positive definite quadratic form

$$V_\rho : x \mapsto x^\mathsf{T} P_0 x, \quad P_0 \in \mathbb{S}^n_{\succ 0} \tag{4}$$

is a Lyapunov function for (1). A such Lyapunov function is often referred to a common Lyapunov function or a parameter-independent Lyapunov function.

Proposition 2.1 (Briat, 2015)**:** *The system* (1) *is quadratically stable if and only if there exists a matrix* $P \in \mathbb{S}^n_{\succ 0}$ *such that the LMI*

$$A^T(\rho)P + PA(\rho) \prec 0 \tag{5}$$

holds for all $\rho \in \mathbf{\Delta}_\rho$.

Moreover, the induced $\mathcal{L}_2$-gain notion is of a big interest for the proposed approach. Indeed, an upper bound is given to the $\mathcal{L}_2$-norm of error between the full-order and the reduced-order models. It can be defined as follows:

Definition 2.2 (Induced $\mathcal{L}_2$-gain; Goddard, 1995)**:** Given a quadratic-stable LPV system (1) with zero initial conditions, the induced $\mathcal{L}_2$-gain is defined as

$$\|G_\rho\|_{i,2} = \sup_{\rho\in\mathcal{P}_\rho} \sup_{\substack{u\in\mathcal{L}_2 \\ u\neq 0}} \frac{\|y\|_2}{\|u\|_2} \tag{6}$$

The induced $\mathcal{L}_2$-norm represents the largest induced gain from inputs in $\mathcal{L}_2$ to outputs in $\mathcal{L}_2$ over the set of all causal linear operators described by the LPV system.

2.2 LPV controller synthesis

Consider a general LPV system represented under the following state-space realisation:

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B_1(\rho(t))w(t) + B_2(\rho(t))u(t) \\ z(t) = C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))u(t) \\ y(t) = C_2(\rho(t))x(t) + D_{21}(\rho(t))w(t) + D_{22}(\rho(t))u(t) \end{cases} \quad \forall t \tag{7}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^{n_u}, w(t) \in \mathbb{R}^{n_w}, z(t) \in \mathbb{R}^{n_z}$ and $y(t) \in \mathbb{R}^{n_y}$ are, respectively, the state, the input, the disturbance, the controlled output and the measured output. Then, $A : \mathbf{\Delta}_\rho \to \mathbb{R}^{n\times n}, B_1 : \mathbf{\Delta}_\rho \to \mathbb{R}^{n\times n_w}, B_2 : \mathbf{\Delta}_\rho \to \mathbb{R}^{n\times n_u}, C_1 : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_z\times n}, C_2 : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_y\times n}, D_{11} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_z\times n_w}, D_{12} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_z\times n_u}, D_{21} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_y\times n_w}$ and $D_{22} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_y\times n_u}$.

The $\mathscr{H}_\infty$-LPV controller associated to the LPV system (7) is defined by

$$K(\rho) : \begin{cases} \dot{x}_\mathrm{K}(t) = A_\mathrm{K}(\rho(t))x_\mathrm{K}(t) + B_\mathrm{K}(\rho(t))y(t) \\ u(t) = C_\mathrm{K}(\rho(t))x_\mathrm{K}(t) + D_\mathrm{K}(\rho(t))y(t) \end{cases} \tag{8}$$

where $x_\mathrm{K} \in \mathbb{R}^{n_\mathrm{K}}, y(t) \in \mathbb{R}^{n_y}$ and $u(t) \in \mathbb{R}^{n_u}$ are, respectively, the states, the inputs and outputs of the controller $K(\rho)$. $A_\mathrm{K} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_\mathrm{K}\times n_\mathrm{K}}, B_\mathrm{K} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_\mathrm{K}\times n_y}, C_\mathrm{K} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_u\times n_\mathrm{K}}$ and $D_\mathrm{K} : \mathbf{\Delta}_\rho \to \mathbb{R}^{n_u\times n_y}$.

The $\mathscr{H}_\infty$-LPV controller synthesis concerns the design of an LPV global controller that guarantees both *stability* and *performance* for all parameter variations defined in the set $\mathbf{\Delta}_\rho$. To guarantee the closed-loop system quadratic stability and to satisfy the $\mathscr{H}_\infty$-performance criteria, the approach developed in Scherer, Gahinet, and Chilali (1997) is used to design the $\mathscr{H}_\infty$-LPV controller. It is assumed that

- the matrices B_2, D_{12}, C_2 and D_{21} are parameter independent and $D_{22} = 0$;
- the parameter-dependent matrix pair $[A(\rho), B_2(\rho)]$ is stabilisable and detectable $\forall\,\rho \in \mathbf{\Delta}_\rho$;
- the matrix $[C_2(\rho), D_{21}(\rho)]$ has full-row rank $\forall\,\rho \in \mathbf{\Delta}_\rho$.

Then, for a given real positive scalar γ and a parameter-dependent quadruplet matrices $(\hat{A}_\mathrm{K}, \hat{B}_\mathrm{K}, \hat{C}_\mathrm{K}, \hat{D}_\mathrm{K})$, there exist two parameter-independent symmetric matrices X and Y such that the sufficient condition that solves the $\mathscr{H}_\infty$-LPV problem is given by the following LMIs $\forall\,\rho \in \mathbf{\Delta}_\rho$:

$$\begin{bmatrix} A(\rho)X + B_2\hat{C}_\mathrm{K}(\rho) + (*) & * & * & * \\ \hat{A}_\mathrm{K}(\rho) + A^\mathsf{T}(\rho) + C_2^\mathsf{T}\hat{D}_\mathrm{K}^\mathsf{T}(\rho)B_2^\mathsf{T} & YA(\rho) + \hat{B}_\mathrm{K}(\rho)C_2 + (*) & * & * \\ B_1^\mathsf{T}(\rho) + D_{21}^\mathsf{T}\hat{D}_\mathrm{K}^\mathsf{T}(\rho)B_2^\mathsf{T} & B_1^\mathsf{T}(\rho)Y + D_{21}^\mathsf{T}\hat{B}_\mathrm{K}^\mathsf{T}(\rho) & -\gamma I_{n_w} & * \\ C_1(\rho)X + D_{12}\hat{C}_\mathrm{K}(\rho) & C_1(\rho) + D_{12}\hat{D}_\mathrm{K}(\rho)C_2 & D_{11}(\rho) + D_{12}\hat{D}_\mathrm{K}(\rho)C_2 & -\gamma I_{n_z} \end{bmatrix} \prec 0 \tag{9}$$

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \succ 0 \tag{10}$$

where terms denoted (*) are induced by symmetry, for example,

$$\begin{bmatrix} M + N + (*) & * \\ K & L \end{bmatrix} = \begin{bmatrix} M + N + M^\mathsf{T} + N^\mathsf{T} & K^\mathsf{T} \\ K & L \end{bmatrix}$$

The LPV controller of the form (8) is then reconstructed as

$$\begin{aligned} D_\mathrm{K}(\rho) &= \hat{D}_\mathrm{K}(\rho) \\ C_\mathrm{K}(\rho) &= (\hat{C}_\mathrm{K}(\rho) - D_\mathrm{K}(\rho)C_2X)M^{-\mathsf{T}} \\ B_\mathrm{K}(\rho) &= N^{-1}(\hat{B}_\mathrm{K}(\rho) - YB_2D_\mathrm{K}(\rho)) \\ A_\mathrm{K}(\rho) &= N^{-1}(\hat{A}_\mathrm{K}(\rho) - YA(\rho)X - YB_2D_\mathrm{K}(\rho)C_2X)M^{-\mathsf{T}} \\ &\quad - B_\mathrm{K}(\rho)C_2XM^{-\mathsf{T}} - N^{-1}YB_2C_\mathrm{K}(\rho) \end{aligned} \tag{11}$$

where M and N are defined such that $MN^\mathsf{T} = I_n - XY$ and which can be solved through a singular value decomposition and a Cholesky factorisation.

2.3 *Full-order controller design*

Before introducing the approximation method, the full-order controller should be designed as shown in Section 2.2. Then, let

$$\begin{aligned} T_{zw}(\rho) &= \left[\begin{array}{c|c} \mathbf{A}(\rho) & \mathbf{B}(\rho) \\ \hline \mathbf{C}(\rho) & \mathbf{D}(\rho) \end{array}\right] \\ &= \mathrm{LFT}\left(\left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right], \left[\begin{array}{c|c} A_\mathrm{K}(\rho) & B_\mathrm{K}(\rho) \\ \hline C_\mathrm{K}(\rho) & D_\mathrm{K}(\rho) \end{array}\right]\right) \end{aligned} \tag{12}$$

be the state-space realisation of the closed-loop system considering the augmented model (7) and its $\mathscr{H}_\infty$-controller defined in (8) with

$$\begin{aligned} \mathbf{A}(\rho) &= \begin{bmatrix} A(\rho) + B_2D_\mathrm{K}(\rho)C_2 & B_2C_\mathrm{K}(\rho) \\ B_\mathrm{K}(\rho)C_2 & A_\mathrm{K}(\rho) \end{bmatrix}, \\ \mathbf{B}(\rho) &= \begin{bmatrix} B_1(\rho) + B_2D_\mathrm{K}(\rho)D_{21} \\ B_\mathrm{K}(\rho)D_{21} \end{bmatrix}, \\ \mathbf{C}(\rho) &= \begin{bmatrix} C_1(\rho) + D_{12}D_\mathrm{K}(\rho)C_2 & D_{12}C_\mathrm{K}(\rho) \end{bmatrix} \text{ and} \\ \mathbf{D}(\rho) &= D_{11}(\rho) + D_{12}D_\mathrm{K}(\rho)D_{21} \end{aligned}$$

Note that the closed-loop system $T_{zw}(\rho)$ in Figure 2 is an LPV system in its own right which is quadratic stable by construction. In order to apply the proposed method, the $T_{zw}(\rho)$ state-space representation is assumed to be minimal.

3. Main contribution

In this section, the new controller order reduction method for LPV controllers is proposed. The method is based on the FWBT approach used for the LTI models order reduction. The FWBT is proposed to reduce the model order over a known frequency range $[\omega_1, \omega_2]$ in Gawronski and Juang (1990). This approach uses the classical reachability and observability Gramians.

Inspired by this work, the proposed method is an extension in two directions:

- extended to the LPV case;
- applied to reduce the controller order (rather than the model order).

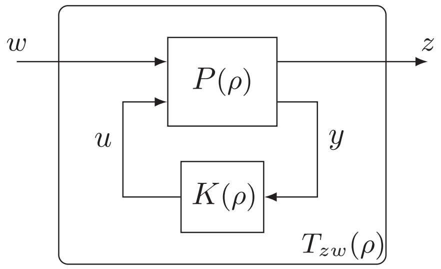

Figure 2. Closed-loop scheme.

To this end, the classical reachability and observability Gramians for LTI systems are introduced first. Then, the equivalent ones of the LPV case are defined. Based on this, the so-called generalised Gramians are introduced and then used to perform the proposed controller order reduction approach. The relation between the Gramians and generalised Gramians is given in Lemma 3.1.

3.1 Frequency-limited generalised gramians

The proposed method is based on the FWBT method where the key notions are the so-called *reachability and observability Gramians.*

Definition 3.1 (Gramians)**:** Considering the quadratic stable and minimal realisation of the closed-loop LPV system $T_{zw}(\rho)$ given in (12), Let us define $\mathcal{R}(\rho)$ and $\mathcal{O}(\rho)$, the related reachability and observability Gramians, respectively; they are defined as the solution of the two parameter-dependent Lyapunov equations

$$-\sum_{i=1}^{s} v_i \frac{\partial \mathcal{R}(\rho)}{\partial \rho_i} + \mathbf{A}(\rho)\mathcal{R}(\rho) + \mathcal{R}(\rho)\mathbf{A}^{\mathsf{T}}(\rho) + \mathbf{B}(\rho)\mathbf{B}^{\mathsf{T}}(\rho) = 0 \tag{13}$$

$$\sum_{i=1}^{s} v_i \frac{\partial \mathcal{O}(\rho)}{\partial \rho_i} + \mathbf{A}^{\mathsf{T}}(\rho)\mathcal{O}(\rho) + \mathcal{O}(\rho)\mathbf{A}(\rho) + \mathbf{C}^{\mathsf{T}}(\rho)\mathbf{C}(\rho) = 0 \tag{14}$$

For a given parameter trajectory ρ, let $\Phi_\rho(t, 0)$ be the state transition matrix. Then, the functional $\mathcal{R}(\rho)$ and $\mathcal{O}(\rho)$, respectively, the reachability and observability Gramians of the closed-loop LPV-system (12), are expressed such that

$$\mathcal{R}(\rho) = \int_{-\infty}^{0} \Phi_\rho(0, t)\mathbf{B}(\rho(t))\mathbf{B}^{\mathsf{T}}(\rho(t))\Phi_\rho^{\mathsf{T}}(0, t)dt \tag{15}$$

$$\mathcal{O}(\rho) = \int_{0}^{\infty} \Phi_\rho^{\mathsf{T}}(t, 0)\mathbf{C}^{\mathsf{T}}(\rho(t))\mathbf{C}(\rho(t))\Phi_\rho(t, 0)dt \tag{16}$$

Definition 3.2 (Generalised Gramians)**:** Let $\mathbf{P}(\rho)$ and $\mathbf{Q}(\rho)$ be, respectively, the *generalised* reachability and observability Gramians of the quadratic stable and minimal closed-loop LPV system (12) given as $\forall(\rho, v) \in \mathbf{\Delta}_\rho \times \mathbf{\Delta}_v$

$$-\sum_{i=1}^{s} v_i \frac{\partial \mathbf{P}(\rho)}{\partial \rho_i} + \mathbf{A}(\rho)\mathbf{P}(\rho) + \mathbf{P}(\rho)\mathbf{A}^{\mathsf{T}}(\rho) + \mathbf{B}(\rho)\mathbf{B}^{\mathsf{T}}(\rho) \prec 0 \tag{17}$$

$$\sum_{i=1}^{s} v_i \frac{\partial \mathbf{Q}(\rho)}{\partial \rho_i} + \mathbf{A}^{\mathsf{T}}(\rho)\mathbf{Q}(\rho) + \mathbf{Q}(\rho)\mathbf{A}(\rho) + \mathbf{C}^{\mathsf{T}}(\rho)\mathbf{C}(\rho) \prec 0 \tag{18}$$

Indeed, Lyapunov inequalities (rather than Lyapunov equations) are solved to compute generalised Gramians. This linear matrix inequality (LMI) approach to the model reduction problem is particularly useful when some structures need to be preserved in the process of model reduction. Controller reduction is a typical example of this type of problems (Shaker & Wisniewski, 2011). Note that the physical interpretations of generalised Gramians are similar to ordinary Gramians. Considering $T_{zw}(\rho)$, $\mathcal{R}(\rho)$ and $\mathcal{O}(\rho)$, the following lemma introduces useful results about the relation between Gramians and generalised Gramians.

Lemma 3.1: *Let $T_{zw}(\rho)$ be a minimal state-space realisation of the quadratic stable and minimal closed-loop system defined by* (12)*. Then, $\forall\rho \in \mathcal{P}_\rho$*

$$\begin{aligned} \mathcal{R}(\rho) &\prec \mathbf{P}(\rho(0)) \\ \mathcal{O}(\rho) &\prec \mathbf{Q}(\rho(0)) \end{aligned} \tag{19}$$

where $\mathcal{R}(\rho)$, $\mathcal{O}(\rho)$, $\mathbf{P}(\rho)$ and $\mathbf{Q}(\rho)$ are the solutions of (15)*,* (16)*,* (17) *and* (18)*, respectively.*

Proof: By multiplying on the left by $\Phi_\rho(0, t)$ and on the right by $\Phi_\rho^{\mathsf{T}}(0, t)$, the relation (17) yields

$$\begin{aligned} &-\Phi_\rho(0,t)\sum_{i=1}^{s} v_i \frac{\partial \mathbf{P}}{\partial \rho_i}\Phi_\rho^{\mathsf{T}}(0,t) + \Phi_\rho(0,t)\mathbf{A}(\rho(t))\mathbf{P}(\rho)\Phi_\rho^{\mathsf{T}}(0,t) \\ &+ \Phi_\rho(0,t)\mathbf{P}(\rho)\mathbf{A}^{\mathsf{T}}(\rho(t))\Phi_\rho^{\mathsf{T}}(0,t) \\ &+ \Phi_\rho(0,t)\mathbf{B}(\rho(t))\mathbf{B}^{\mathsf{T}}(\rho(t))\Phi_\rho^{\mathsf{T}}(0,t) \prec 0 \end{aligned} \tag{20}$$

Using the fact that $\frac{\partial}{\partial t_0}\Phi_\rho(t,t_0) = -\Phi_\rho(t,t_0)A(\rho(t_0))$, then (20) yields

$$-\frac{d}{dt}\left(\Phi_\rho(0,t)\mathbf{P}(\rho)\Phi_\rho^{\mathsf{T}}(0,t)\right) + \Phi_\rho(0,t)\mathbf{B}(\rho(t))\mathbf{B}^{\mathsf{T}}(\rho(t))\Phi_\rho^{\mathsf{T}}(0,t) \prec 0 \quad (21)$$

which on integrating over the semi-infinite time axis $(-\infty, 0]$ and considering that $\lim_{t\to-\infty}\Phi_\rho(0,t) = 0$, gives

$$\mathbf{P}(\rho(0)) \succ \underbrace{\int_{-\infty}^{0} \Phi_\rho(0,t)\mathbf{B}(\rho(t))\mathbf{B}^{\mathsf{T}}(\rho(t))\Phi_\rho^{\mathsf{T}}(0,t)dt}_{\mathcal{R}(\rho)} \quad (22)$$

Similarly, let us multiply on the right by the closed-loop state transition matrix $\Phi_\rho(t,0)$ and on the left by $\Phi_\rho^{\mathsf{T}}(t,0)$, the relation (18)

$$\begin{aligned}&\Phi_\rho^{\mathsf{T}}(t,0)\sum_{i=1}^{s} \upsilon_i \frac{\partial \mathbf{Q}}{\partial \rho_i}\Phi_\rho(t,0) + \Phi_\rho^{\mathsf{T}}(t,0)\mathbf{A}^{\mathsf{T}}(\rho(t))\mathbf{Q}(\rho)\Phi_\rho(t,0)\\ &+\Phi_\rho^{\mathsf{T}}(t,0)\mathbf{Q}(\rho)\mathbf{A}(\rho(t))\Phi_\rho(t,0)\\ &+\Phi_\rho^{\mathsf{T}}(t,0)\mathbf{C}^{\mathsf{T}}(\rho(t))\mathbf{C}(\rho(t))\Phi_\rho(t,0) \prec 0 \quad (23)\end{aligned}$$

Using the fact that $\frac{\partial}{\partial t}\Phi_\rho(t,t_0) = A(\rho(t))\Phi_\rho(t,t_0)$, then (23) yields

$$\frac{d}{dt}\left(\Phi_\rho^{\mathsf{T}}(t,0)\mathbf{Q}(\rho)\Phi_\rho(t,0)\right) + \Phi_\rho^{\mathsf{T}}(t,0)\mathbf{C}^{\mathsf{T}}(\rho(t))\mathbf{C}(\rho(t))\Phi_\rho(t,0) \prec 0 \quad (24)$$

Integrating the last expression from 0 to $+\infty$ gives

$$\mathbf{Q}(\rho(0)) \succ \underbrace{\int_{0}^{+\infty} \Phi_\rho^{\mathsf{T}}(t,0)\mathbf{C}^{\mathsf{T}}(\rho(t))\mathbf{C}(\rho(t))\Phi_\rho(t,0)dt}_{\mathcal{O}(\rho)} \quad (25)$$

■

3.2 Frequency-limited order reduction

For a given parameter trajectory ρ, let consider $\Phi_\rho(t,0)$ the state transition matrix of the closed-loop system (12). Then, let us define

$$\begin{aligned} f_\rho(t) &= \Phi_\rho(0,t)\mathbf{B}(\rho(t))H(-t) \\ g_\rho(t) &= \mathbf{C}(\rho(t))\Phi_\rho(t,0)H(t) \end{aligned} \quad \forall t \quad (26)$$

where H is the Heaviside step function.

By considering $\Omega = [\omega_1;\omega_2]$ the frequency range where order reduction is desired to be better, the following definition is given.

Definition 3.3 (Frequency-limited Gramians): Let $\mathcal{R}_\Omega(\rho)$ and $\mathcal{O}_\Omega(\rho)$, two frequency-dependent terms, defined as

$$\mathcal{R}_\Omega(\rho) = \mathcal{R}_{\omega_2}(\rho) - \mathcal{R}_{\omega_1}(\rho) \quad \text{and} \quad \mathcal{O}_\Omega(\rho) = \mathcal{O}_{\omega_2}(\rho) - \mathcal{O}_{\omega_1}(\rho)$$

where

$$\mathcal{R}_\omega(\rho) = \int_{-\omega}^{+\omega} F_\rho(w)F_\rho^*(w)dw \quad (27)$$

$$\mathcal{O}_\omega(\rho) = \int_{-\omega}^{+\omega} G_\rho^*(w)G_\rho(w)dw \quad (28)$$

with F_ρ (respectively G_ρ) is the Fourier transform of f_ρ (respectively g_ρ). Then, the functional $\hat{\mathcal{R}}_\Omega(\rho)$ and $\hat{\mathcal{O}}_\Omega(\rho)$, respectively, the *frequency-limited* reachability and observability Gramians of the closed-loop LPV system (12), are defined as the solutions of $\forall(\rho, v) \in \mathbf{\Delta}_\rho \times \mathbf{\Delta}_v$

$$-\sum_{i=1}^{s} \upsilon_i \frac{\partial \hat{\mathcal{R}}_\Omega(\rho)}{\partial \rho_i} + \mathbf{A}(\rho)\hat{\mathcal{R}}_\Omega(\rho) + \hat{\mathcal{R}}_\Omega(\rho)\mathbf{A}^{\mathsf{T}}(\rho) + \mathcal{R}_\Omega(\rho) = 0 \quad (29)$$

$$\sum_{i=1}^{s} \upsilon_i \frac{\partial \hat{\mathcal{O}}_\Omega(\rho)}{\partial \rho_i} + \mathbf{A}^{\mathsf{T}}(\rho)\hat{\mathcal{O}}_\Omega(\rho) + \hat{\mathcal{O}}_\Omega(\rho)\mathbf{A}(\rho) + \mathcal{O}_\Omega(\rho) = 0 \quad (30)$$

The quantities $\mathcal{R}_\Omega(\rho)$ and $\mathcal{O}_\Omega(\rho)$ have the following eigenvalues decomposition:

$$\mathcal{R}_\Omega(\rho) = U_\Omega(\rho)\mathrm{diag}\left(\lambda_1(\rho), \ldots, \lambda_{n_{\mathrm{K}}}(\rho)\right)U_\Omega^{\mathsf{T}}(\rho) \quad (31)$$

$$\mathcal{O}_\Omega(\rho) = V_\Omega(\rho)\mathrm{diag}\left(\delta_1(\rho), \ldots, \delta_{n_{\mathrm{K}}}(\rho)\right)V_\Omega^{\mathsf{T}}(\rho) \quad (32)$$

with $|\lambda_1| \geq \cdots |\lambda_{n_{\mathrm{K}}}| \geq 0$ and $|\delta_1| \geq \cdots |\delta_{n_{\mathrm{K}}}| \geq 0$. Let $u_\rho \leq n_{\mathrm{K}}$ and $v_\rho \leq n_{\mathrm{K}}$ be, respectively, the ranks of $\mathcal{R}_\Omega(\rho)$ and $\mathcal{O}_\Omega(\rho)$. Based on these definitions, let us define the two quantities

$$\mathbf{B}_\Omega(\rho) = U_\Omega(\rho)\mathrm{diag}(|\lambda_1(\rho)|^{\frac{1}{2}}, \ldots, |\lambda_{u_\rho}(\rho)|^{\frac{1}{2}}, 0, \ldots, 0), \text{ and} \quad (33)$$

$$\mathbf{C}_\Omega(\rho) = \mathrm{diag}(|\delta_1(\rho)|^{\frac{1}{2}}, \ldots, |\delta_{\upsilon_\rho}(\rho)|^{\frac{1}{2}}, 0, \ldots, 0)V_\Omega^{\mathsf{T}}(\rho) \quad (34)$$

Definition 3.4 (Modified frequency-limited generalised Gramians)**:** Consider $K(\rho)$, the full-order stabilising LPV controller given in (8). Let $\hat{\mathbf{P}}_\Omega(\rho) = \left[\begin{smallmatrix} \hat{\mathbf{P}}_1(\rho) & 0 \\ 0 & \hat{\mathbf{P}}_2(\rho) \end{smallmatrix}\right] \succ 0$ and $\hat{\mathbf{Q}}_\Omega(\rho) = \left[\begin{smallmatrix} \hat{\mathbf{Q}}_1(\rho) & 0 \\ 0 & \hat{\mathbf{Q}}_2(\rho) \end{smallmatrix}\right] \succ 0$ be the *modified frequency-limited* reachability and observability Gramians defined as the solutions of the following Lyapunov equations $\forall(\rho, v) \in \mathbf{\Delta}_\rho \times \mathbf{\Delta}_v$:

$$
\begin{aligned}
-\sum_{i=1}^{s} v_i \frac{\partial \hat{\mathbf{P}}_\Omega(\rho)}{\partial \rho_i} &+ \mathbf{A}(\rho)\hat{\mathbf{P}}_\Omega(\rho) + \hat{\mathbf{P}}_\Omega(\rho)\mathbf{A}^{\mathsf{T}}(\rho) \\
&+ \mathbf{B}_\Omega(\rho)\mathbf{B}_\Omega^{\mathsf{T}}(\rho) = 0
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\sum_{i=1}^{s} v_i \frac{\partial \hat{\mathbf{Q}}_\Omega(\rho)}{\partial \rho_i} &+ \mathbf{A}^{\mathsf{T}}(\rho)\hat{\mathbf{Q}}_\Omega(\rho) + \hat{\mathbf{Q}}_\Omega(\rho)\mathbf{A}(\rho) \\
&+ \mathbf{C}_\Omega^{\mathsf{T}}(\rho)\mathbf{C}_\Omega(\rho) = 0.
\end{aligned} \tag{36}
$$

For the generalisation, we have the following inequalities:

$$
\begin{aligned}
-\sum_{i=1}^{s} v_i \frac{\partial \mathbf{P}_\Omega(\rho)}{\partial \rho_i} &+ \mathbf{A}(\rho)\mathbf{P}_\Omega(\rho) + \mathbf{P}_\Omega(\rho)\mathbf{A}^{\mathsf{T}}(\rho) \\
&+ \mathbf{B}_\Omega(\rho)\mathbf{B}_\Omega^{\mathsf{T}}(\rho) \prec 0
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\sum_{i=1}^{s} v_i \frac{\partial \mathbf{Q}_\Omega(\rho)}{\partial \rho_i} &+ \mathbf{A}^{\mathsf{T}}(\rho)\mathbf{Q}_\Omega(\rho) + \mathbf{Q}_\Omega(\rho)\mathbf{A}(\rho) \\
&+ \mathbf{C}_\Omega^{\mathsf{T}}(\rho)\mathbf{C}_\Omega(\rho) \prec 0.
\end{aligned} \tag{38}
$$

with $\mathbf{P}_\Omega(\rho) = \left[\begin{smallmatrix} \mathbf{P}_1(\rho) & 0 \\ 0 & \mathbf{P}_2(\rho) \end{smallmatrix}\right] \succ 0$ and $\mathbf{Q}_\Omega(\rho) = \left[\begin{smallmatrix} \mathbf{Q}_1(\rho) & 0 \\ 0 & \mathbf{Q}_2(\rho) \end{smallmatrix}\right] \succ 0$.

If the block diagonal solutions $\mathbf{P}_\Omega(\rho)$ and $\mathbf{Q}_\Omega(\rho)$ exist, then let $T_1(\rho)$ and $T_2(\rho)$ be two non-singular matrices given such that

$$
\begin{aligned}
T_1^{-1}(\rho)\mathbf{P}_1(\rho)T_1^{-\mathsf{T}}(\rho) &= T_1^{\mathsf{T}}(\rho)\mathbf{Q}_1(\rho)T_1(\rho) \\
&= \Sigma_1(\rho) \\
&= \mathrm{diag}(\xi_1(\rho), \ldots, \xi_n(\rho)),
\end{aligned} \tag{39}
$$

with $\xi_1(\rho) \geq \xi_2(\rho) \geq \ldots \geq \xi_n(\rho)$, and

$$
\begin{aligned}
&T_2^{-1}(\rho)\mathbf{P}_2(\rho)T_2^{-\mathsf{T}}(\rho) \\
&= T_2^{\mathsf{T}}(\rho)\mathbf{Q}_2(\rho)T_2(\rho) \\
&= \Sigma_2(\rho) \\
&= \mathrm{diag}(\underbrace{\gamma_1(\rho), \ldots, \gamma_r(\rho)}_{\Sigma_{21}}, \underbrace{\gamma_{r+1}(\rho), \ldots, \gamma_{n_{\mathrm{K}}}(\rho)}_{\Sigma_{22}})
\end{aligned} \tag{40}
$$

with $\gamma_1(\rho) \geq \gamma_2(\rho) \geq \cdots \geq \gamma_r(\rho) > \gamma_{r+1}(\rho) \geq \gamma_{r+2}(\rho) \geq \cdots \geq \gamma_{n_{\mathrm{K}}}(\rho)$ are the frequency-limited generalised Hankel singular values of $K(\rho)$ and r is the desired order for the reduced-order controller.

The balanced realisation of $K(\rho)$ can be written as

$$
\begin{aligned}
\tilde{K}(\rho) &= \left[\begin{array}{c|c} T_2^{-1}(\rho)A_{\mathrm{K}}(\rho)T_2(\rho) & T_2^{-1}(\rho)B_{\mathrm{K}}(\rho) \\ \hline C_{\mathrm{K}}(\rho)T_2(\rho) & D_{\mathrm{K}}(\rho) \end{array}\right] \\
&= \left[\begin{array}{c|c} \tilde{A}_{\mathrm{K}}(\rho) & \tilde{B}_{\mathrm{K}}(\rho) \\ \hline \tilde{C}_{\mathrm{K}}(\rho) & \tilde{D}_{\mathrm{K}}(\rho) \end{array}\right].
\end{aligned} \tag{41}
$$

Further, $\tilde{K}(\rho)$ is partitioned as conformably with $\Sigma_2(\rho)$ as

$$
\tilde{K}(\rho) = \left[\begin{array}{c|c} \begin{bmatrix} \hat{A}_{\mathrm{K}}(\rho) & \tilde{A}_{\mathrm{K}12}(\rho) \\ \tilde{A}_{\mathrm{K}21}(\rho) & \tilde{A}_{\mathrm{K}22}(\rho) \end{bmatrix} & \begin{bmatrix} \hat{B}_{\mathrm{K}}(\rho) \\ \tilde{B}_{\mathrm{K}2}(\rho) \end{bmatrix} \\ \hline \begin{bmatrix} \hat{C}_{\mathrm{K}}(\rho) & \tilde{C}_{\mathrm{K}2}(\rho) \end{bmatrix} & \hat{D}_{\mathrm{K}}(\rho) \end{array}\right]. \tag{42}
$$

Finally, a truncation step is performed to obtain a reduced-order controller.

Definition 3.5: Given the balanced realisation $\tilde{K}(\rho)$ defined in (42), let $\hat{K}(\rho)$ be the truncated realisation to the rth order and denoted as follows:

$$
\hat{K}(\rho) = \left[\begin{array}{c|c} \hat{A}_{\mathrm{K}}(\rho) & \hat{B}_{\mathrm{K}}(\rho) \\ \hline \hat{C}_{\mathrm{K}}(\rho) & \hat{D}_{\mathrm{K}}(\rho) \end{array}\right] \tag{43}
$$

Furthermore, the reduced-order parameter-dependent closed-loop system is given as

$$
\begin{aligned}
\hat{T}_{zw}(\rho) &= \left[\begin{array}{c|c} \hat{\mathbf{A}}(\rho) & \hat{\mathbf{B}}(\rho) \\ \hline \hat{\mathbf{C}}(\rho) & \hat{\mathbf{D}}(\rho) \end{array}\right] \\
&= \left[\begin{array}{cc|c} A(\rho) + B_2\hat{D}_{\mathrm{K}}(\rho)C_2 & B_2\hat{C}_{\mathrm{K}}(\rho) & B_1(\rho) + B_2\hat{D}_{\mathrm{K}}(\rho)\hat{D}_{\mathrm{K}}(\rho)D_{21} \\ \hat{B}_{\mathrm{K}}(\rho)C_2 & \hat{A}_{\mathrm{K}}(\rho) & \hat{B}_{\mathrm{K}}(\rho)D_{21} \\ \hline C_1(\rho) + D_{12}\hat{D}_{\mathrm{K}}(\rho)C_2 & D_{12}\hat{C}_{\mathrm{K}}(\rho) & D_{11}(\rho) + D_{12}\hat{D}_{\mathrm{K}}(\rho)D_{21} \end{array}\right]
\end{aligned} \tag{44}
$$

Theorem 3.1: *Suppose $K(\rho)$ is the stabilising parameter-dependent controller defined in (8) such that the closed-loop transfer $T_{zw}(\rho)$ defined in (12) is minimal, quadratic stable, and there exist Lyapunov inequality solutions $\mathbf{P}_\Omega(\rho)$ and $\mathbf{Q}_\Omega(\rho)$ such that (37) and (38) are satisfied. Let $\hat{K}(\rho)$ be the reduced-order controller defined in (43) and obtained by truncation. Then, the closed-loop system with the reduced-order controller $\hat{T}_{zw}(\rho)$ defined in (44) is stable. If, in addition,*

$$\operatorname{rank}[\mathbf{B}(\rho), \mathbf{B}_\Omega(\rho)] = \operatorname{rank}[\mathbf{B}_\Omega(\rho)] \tag{45}$$

and

$$\operatorname{rank}[\mathbf{C}^T(\rho), \mathbf{C}_\Omega^T(\rho)] = \operatorname{rank}[\mathbf{C}_\Omega^T(\rho)] \tag{46}$$

then, $\hat{T}_{zw}(\rho)$ is quadratic stable and satisfies

$$\|T_{zw}(\rho) - \hat{T}_{zw}(\rho)\|_{i,2} \le 2\|J_B(\rho)\|_\infty \|J_C(\rho)\|_\infty \sum_{i=r+1}^{n_K} \gamma_{i,\rho} \tag{47}$$

where $J_B(\rho) := \operatorname{diag}(|\lambda_1|^{-\frac{1}{2}}(\rho), \ldots, |\lambda_{u_\rho}|^{-\frac{1}{2}}(\rho), 0, \ldots, 0)$ $U_\Omega(\rho)\mathbf{B}(\rho)$ *and*

$J_C(\rho) := \mathbf{C}(\rho)V_\Omega(\rho)\operatorname{diag}(|\delta_1|^{-\frac{1}{2}}(\rho), \ldots, |\delta_{v_\rho}|^{-\frac{1}{2}}(\rho), 0, \ldots, 0)$.

Proof: The reachability and the observability Gramians given in (15) and (16) can be expressed as

$$\mathcal{R}(\rho) = \int_{-\infty}^{+\infty} f_\rho(\tau) f_\rho^*(\tau) d\tau \tag{48}$$

$$\mathcal{O}(\rho) = \int_{-\infty}^{\infty} g_\rho^*(\tau) g_\rho(\tau) d\tau \tag{49}$$

where $f_\rho(\tau)$ and $g_\rho(\tau)$ are given in (26).

Then, using Parseval relationship, the reachability and the observability Gramians could be expressed as follows:

$$\mathcal{R}(\rho) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_\rho(w) F_\rho^*(w) dw \tag{50}$$

$$\mathcal{O}(\rho) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\rho^*(w) G_\rho(w) dw \tag{51}$$

By considering F_ρ(resp.g_ρ), the Fourier transform of f_ρ(respectively g_ρ), we can define $\mathcal{R}_\omega(\rho)$ and $\mathcal{O}_\omega(\rho)$ as the limited reachability and observability Gramians given in (27) and (28). Then, by setting $\Omega = [\omega_1 ; \omega_2]$ the frequency range where approximation is desired to be better. Then, new modified terms $\mathcal{R}_\Omega(\rho)$ and $\mathcal{O}_\Omega(\rho)$ are defined.

Since $\mathcal{R}_\Omega(\rho)$ and $\mathcal{O}_\Omega(\rho)$ are not guaranteed to be positive definite, the stability of the reduced-order controller is not guaranteed. Then, an idea based on eigenvalues decomposition presented in Gugercin and Antoulas (2004) is proposed here to guarantee the stability by providing an upper error bound. Indeed, the solution of (37) and (38) is performed instead of (29) and (30). In addition, the modified frequency-limited Gramians in (35) and (36) are used instead of the ordinary ones defined in (35) and (36). In fact, Lemma 3.1 shows that the generalised Gramians could be an approximation of the ordinary Gramians. The transition to generalised Gramian framework might induce less accurate approximation but the order-reduction error still bounded. Finally, the expression of the upper error bound given in (47) is found according to the following.

Let $\tilde{T}_{zw}(\rho)$ be the frequency-limited balanced realisation of the full-order closed-loop system. Then, $\tilde{T}_{zw}(\rho)$ is defined by

$$\tilde{T}_{zw}(\rho) = \left[\begin{array}{c|c} T^{-1}(\rho)\mathbf{A}(\rho)T(\rho) & T^{-1}(\rho)\mathbf{B}(\rho) \\ \hline \mathbf{C}(\rho)T(\rho) & \mathbf{D}(\rho) \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} \hat{\mathbf{A}}(\rho) & \tilde{\mathbf{A}}_{12}(\rho) \\ \tilde{\mathbf{A}}_{21}(\rho) & \tilde{\mathbf{A}}_{22}(\rho) \end{bmatrix} & \begin{bmatrix} \hat{\mathbf{B}}(\rho) \\ \tilde{\mathbf{B}}_2(\rho) \end{bmatrix} \\ \hline \begin{bmatrix} \hat{\mathbf{C}}(\rho) & \tilde{\mathbf{C}}_2(\rho) \end{bmatrix} & \tilde{\mathbf{D}}(\rho) \end{array}\right] \tag{52}$$

where $T(\rho) = \operatorname{diag}(T_1(\rho), T_2(\rho))$.

By considering the assumptions (45) and (46), there exist $J_B(\rho)$ and $J_C(\rho)$ such that $\mathbf{B}(\rho) = \mathbf{B}_\Omega(\rho)J_B(\rho)$ and $\mathbf{C}(\rho) = J_C(\rho)\mathbf{C}_\Omega(\rho)$. On the other hand, the reduction error is expressed as

$$\begin{aligned} \|T_{zw}(\rho) - \hat{T}_{zw}(\rho)\|_{i,2} &= \|\mathbf{C}(\rho)(sI - \mathbf{A}(\rho))^{-1}\mathbf{B}(\rho) \\ &\quad -\hat{\mathbf{C}}(\rho)(sI - \hat{\mathbf{A}}(\rho))^{-1}\hat{\mathbf{B}}(\rho)\|_{i,2} \\ &= \|J_C(\rho)[\mathbf{C}_\Omega(\rho)(sI - \mathbf{A}(\rho))^{-1}\mathbf{B}_\Omega(\rho) \\ &\quad -\hat{\mathbf{C}}(\rho)(sI - \hat{\mathbf{A}}(\rho))^{-1}\hat{\mathbf{B}}(\rho)].J_B(\rho)\|_{i,2} \\ &\le 2\|J_B(\rho)\|_\infty \|J_C(\rho)\|_\infty \sum_{i=r+1}^{n_K} \gamma_{i,\rho} \end{aligned}$$

■

Remark 3.1: Assumptions (45) and (46) mean that there exist $J_B(\rho)$ and $J_C(\rho)$ such that $\mathbf{B}(\rho) = \mathbf{B}_\Omega(\rho)J_B(\rho)$ and $\mathbf{C}(\rho) = J_C(\rho)\mathbf{C}_\Omega(\rho)$. In addition, by following the steps in Wang et al. (1999), Anderson and Clements (1981) and Imran, Ghafoor, and Sreeram (2014), it was shown that Assumptions (45) and (46) are almost always true. Hence we expect that our approach will apply in most of the cases. Indeed, during our simulations, the assumptions have always been satisfied.

Algorithm 1 $\mathscr{H}_\infty$-LPV controller order reduction

Considering the LPV plant described in (7), the reduced-order controller can be computed as follows:

Inputs: $(A(\rho), B(\rho), C(\rho), D(\rho))$.

Outputs: $(\hat{A}(\rho), \hat{B}(\rho), \hat{C}(\rho), \hat{D}(\rho))$.

Assumptions: $(\mathbf{A}(\rho), \mathbf{B}(\rho), \mathbf{C}(\rho), \mathbf{D}(\rho))$ minimal.

Algorithm:

1. Compute $K(\rho)$ the full-order controller according to Section 2.2.
2. Compute the closed-loop system T_{zw} given in (12).
3. Compute the generalised Gramian $\mathbf{P}(\rho)$ and $\mathbf{Q}(\rho)$ solutions of (17) and (18), respectively.
4. Compute the balanced realisation $\tilde{K}(\rho)$ of the full-order controller $K(\rho)$ by the following:
   - (a) Find T_2, the basis change matrix according to (40) (for instance, Procedure in []Moore1988 can be used).
   - (b) Compute the balanced realisation $\tilde{K}(\rho)$ as defined in (42).
5. Compute the reduced-order controller $\hat{K}(\rho)$ from $\tilde{K}(\rho)$ by truncation.

4. Case study: semi-active suspension control

4.1 LPV model

In this section, the given method to reduce the LPV controller is applied on a semi-active automotive suspension presented in Do, Sename, Dugard, and Boussaad (2011). Actually, when suspension modelling and control are considered, the vertical quarter car model is often used. This model allows to study the vertical behaviour of a vehicle according to the suspension characteristic (passive or controlled). Figure 3 shows the so-called vertical quarter car. Then, the dynamical equations of the system are given by

$$\begin{cases} m_s\ddot{z}_s = -k_s z_{\text{def}} - F_{\text{mr}} \\ m_{\text{us}}\ddot{z}_{\text{us}} = k_s z_{\text{def}} + F_{\text{mr}} - k_t(z_{\text{us}} - z_r) \end{cases} \tag{53}$$

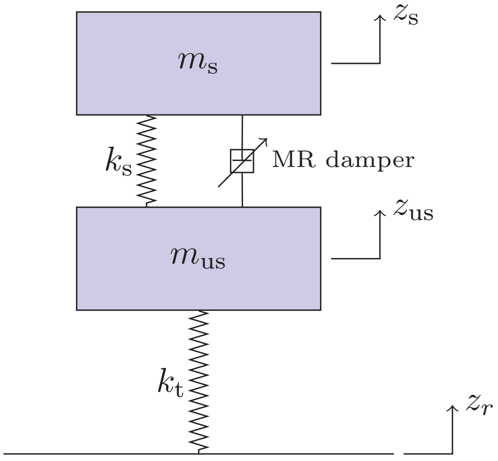

Figure 3. Quarter-vehicle model.

where F_{mr} is the magneto-rheological force generated by the semi-active suspension. According to the nonlinear model of Guo, Yang, and Pan (2006), F_{mr} can be expressed as follows:

$$F_{\text{mr}} = a_2\left(\dot{z}_{\text{def}} + \frac{v_0}{x_0}z_{\text{def}}\right) + a_1\tanh\left(a_3\left(\dot{z}_{\text{def}} + \frac{v_0}{x_0}z_{\text{def}}\right)\right) \tag{54}$$

with $z_{\text{def}} = z_s - z_{\text{us}}$ is the damper deflection (must be measured or estimated) and $\dot{z}_{\text{def}} = \dot{z}_s - \dot{z}_{\text{us}}$ is the damper velocity. Parameters a_2, a_3, v_0 and x_0 are constant, and a_1 is the controllable force such that $a_1 \in [a_{1\min}\,; a_{1\max}]$.

By defining
$\rho_1 = \tanh\left(a_3\left(\dot{z}_{\text{def}} + \frac{v_0}{x_0}z_{\text{def}}\right)\right)$,
$c_{\text{mr}} = a_2$: MR damping coefficient,
$k_{\text{mr}} = a_2\frac{v_0}{x_0}$: MR stiffness coefficient,

Then, a state-space representation can be given by considering the state vector $x_s = [z_s\dot{z}_s z_{\text{us}}\dot{z}_{\text{us}}]^{\mathsf{T}}$ and the exogenous input $w = z_r$, as follows:

$$\begin{cases} \dot{x}_s = A_s x_s + B_s\rho_1 a_1 + B_{s1}w \\ z = C_{s1}x_s + D_{s1}\rho_1 a_1 \\ y = C_s x_s \end{cases} \tag{55}$$

with

$$A_s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s+k_{\text{mr}}}{m_s} & -\frac{c_{\text{mr}}}{m_s} & \frac{k_s+k_{\text{mr}}}{m_s} & \frac{c_{\text{mr}}}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s+k_{\text{mr}}}{m_{\text{us}}} & \frac{c_{\text{mr}}}{m_{\text{us}}} & -\frac{k_s+k_{\text{mr}}+k_t}{m_{\text{us}}} & -\frac{c_{\text{mr}}}{m_{\text{us}}} \end{pmatrix},$$

$$B_s = \begin{pmatrix} 0 \\ \frac{-1}{m_s} \\ 0 \\ \frac{-1}{m_{\text{us}}} \end{pmatrix}, B_{s1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-k_t}{m_{\text{us}}} \end{pmatrix}.$$

The measurement output is $y = z_s - z_{\text{us}}$, and the controlled outputs are chosen as $z = [\ddot{z}_s \quad z_s]^{\mathsf{T}}$, respectively, the acceleration and the displacement of the sprung mass. Then

$$C_{s1} = \begin{pmatrix} -\frac{k_s+k_{\text{mr}}}{m_s} & -\frac{c_{\text{mr}}}{m_s} & \frac{k_s+k_{\text{mr}}}{m_s} & \frac{c_{\text{mr}}}{m_s} \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$D_{s1} = \begin{pmatrix} -\frac{1}{m_s} \\ 0 \end{pmatrix}, C_s = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}.$$

However, two constraints must be satisfied:

(1) The control signal a_1 must be positive (dissipative constraint).
(2) The input matrices $B_s\rho_1$ and $D_{s1}\rho_1$ must be constant to satisfy the LPV-$\mathscr{H}_\infty$ synthesis assumption.

The passivity problem is solved by defining a new control signal $u = a_1 - F_0$, where F_0 is the mean value of a_1 ($F_0 = (a_{1\max} - a_{1\min})/2$). Then, the problem of the passivity on a_1 is recast to a simple saturation problem on u ($u \in [-F_0\,; F_0]$). With these modifications, (56) yields

$$\begin{cases} \dot{x}_s = (A_s + B_{s2}\dfrac{\rho_1}{C_{s2}x_s}C_{s2})x_s + B_s\rho_1 u + B_{s1}w \\ z = C_{s1}x_s + D_{s1}\rho_1 u \\ y = C_s x_s \end{cases} \tag{56}$$

where

$$B_{s2} = \left(0 \; -\tfrac{F_0}{m_s} \; 0 \; \tfrac{F_0}{m_{us}}\right)^{\mathsf{T}} \text{ and } C_{s2} = \left(-\tfrac{a_3 v_0}{x_0} \;\; a_3 \; -\tfrac{a_3 v_0}{x_0} \;\; -a_3\right)^{\mathsf{T}}$$

To overcome the second problem, Apkarian and Gahinet (1995) propose to add a strictly proper filter $\mathcal{F}$ to make the controlled output matrices independent of the scheduling parameters

$$\mathcal{F} : \begin{pmatrix} \dot{x}_f \\ u \end{pmatrix} = \begin{pmatrix} A_f & B_f \\ C_f & 0 \end{pmatrix} \begin{pmatrix} x_f \\ u_c \end{pmatrix} \tag{57}$$

Then, by defining $\rho_2 = \frac{\rho_1}{C_{s2}x_s}$ and $x = (x_s \quad x_f)^{\mathsf{T}}$, the system (56) can be represented as

$$\begin{cases} \dot{x} = A(\rho_1, \rho_2)x + Bu_c + B_1 w \\ z = C_1(\rho_1, \rho_2)x \\ y = Cx \end{cases} \tag{58}$$

where

$$A(\rho_1, \rho_2) = \begin{pmatrix} A_s + \rho_2 B_{s2}C_{s2} & \rho_1 B_s C_f \\ 0 & A_f \end{pmatrix},$$
$$B = \begin{pmatrix} 0 \\ B_f \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_{s1} \\ 0 \end{pmatrix},$$
$$C_1(\rho_1, \rho_2) = \begin{pmatrix} C_{s1} & \rho_1 D_{s1} C_f \end{pmatrix} \text{ and } C = \begin{pmatrix} C_s & 0 \end{pmatrix}$$

4.2 $\mathscr{H}_\infty$-controller synthesis

By considering the LPV model (58), an $\mathscr{H}_\infty$-controller is designed to guarantee the internal closed-loop stability and to satisfy some required performance. In fact, the main objective and challenge of a controlled suspension system is to improve the comfort for car passengers simultaneously to the performance on road holding. The passenger comfort can be improved by isolating the vibrations transmitted from the road surface. Then, the 'frequency response' from the road profile z_r to the vehicle chassis acceleration $\ddot{z}_s$ must be kept small in the low-frequency range. Then, a weighting function is designed as

$$W_{\ddot{z}_s} = w_{\ddot{z}_s} \frac{s^2 + \xi_{11} w_{11} s + w_{11}^2}{s^2 + 2\xi_{12} w_{12} s + w_{12}^2}$$

Furthermore, the road holding is evaluated from the unsprung mass (wheel) oscillations with respect to the road profile. This transfer should be kept small at high frequencies. Then, $W_{z_{us}}$ is designed as

$$W_{z_{us}} = w_{z_{us}} \frac{s^2 + \xi_{21} w_{21} s + w_{21}^2}{s^2 + 2\xi_{22} w_{22} s + w_{22}^2}$$

$W_{z_r} = 5 \times 10^{-3}$ is the road profile gain. Finally, the filter introduced in (57) is given as $\mathcal{F} = \frac{w_f}{s + w_f}$. It is designed with a large bandwidth to decouple the input and the varying parameters, where

$$w_{\ddot{z}_s} = 1, \xi_{11} = 0.1, \xi_{12} = 1, w_{11} = 2\pi \times 1\,\text{rad.s}^{-1},$$
$$w_{12} = 2\pi \times 3\,\text{rads}^{-1}$$

$$w_{z_{us}} = 10, \xi_{21} = 0.3, \xi_{22} = 1, w_{21} = 2\pi \times 9\,\text{rads}^{-1},$$
$$w_{22} = 2\pi \times 9\,\text{rads}^{-1}, w_f = 90.34$$

Then, an interconnection between the LPV model and these weighting functions are presented in Figure 4.

Model parameters are obtained by considering experimental data in Tudon-Martınez et al. (2013) and given as Table 1.

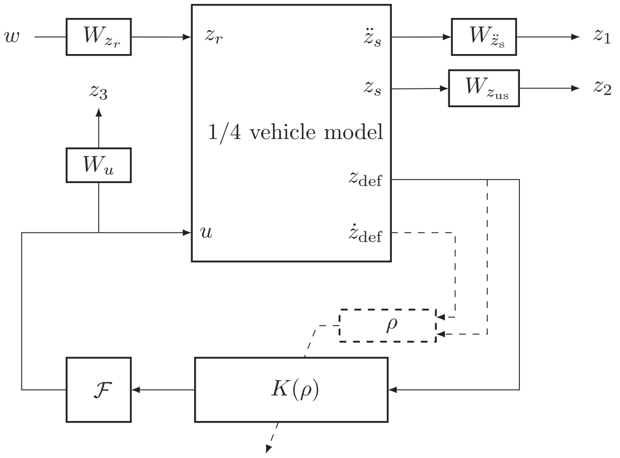

Figure 4. $\mathscr{H}_\infty$−control scheme.

Table 1. Values of $\left(\text{rank}[\mathbf{B}(\rho), \mathbf{B}_\Omega(\rho)]/\text{rank}[\mathbf{B}_\Omega(\rho)]\,;\ \text{rank}[\mathbf{C}^\mathrm{T}(\rho), \mathbf{C}_\Omega^\mathrm{T}(\rho)]/\text{rank}[\mathbf{C}_\Omega^\mathrm{T}(\rho)]\right)$ for the several frozen values of (ρ_1, ρ_2).

ρ_1/ρ_2	−1.0	−0.5	0.0	0.5	1.0
0	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)
0.25	(14/14 ; 14/14)	(13/13 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)
0.5	(11/11 ; 14/14)	(13/13 ; 14/14)	(11/11 ; 14/14)	(12/12 ; 14/14)	(11/11 ; 14/14)
0.75	(12/12 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)
1.0	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)	(14/14 ; 14/14)

To carry out a controller satisfying these objectives, the $\mathscr{H}_\infty$-LPV synthesis is designed by using solution for polytopic systems: it consists in finding a global LPV controller $K(\rho_1, \rho_2)$ which is a convex combination of local controllers obtained by solving the LMIs set at each vertex (formed by limits values of the varying parameters). All varying parameters are bounded: $\rho_1 \in [-1\,;1]$, $\rho_2 \in [0\,;1]$.

For more details and explanation on $\mathscr{H}_\infty$-LPV control synthesis, see Apkarian and Gahinet (1995) and Scherer et al. (1997). The design method for LPV systems is used like in Apkarian, Gahinet, and Becker (1994).

4.3 Numerical issue

The proposed method requests the solution of two Lyapunov inequalities with an infinite number of constraints. These sets of infinite LMIs can be solved by gridding techniques. Then, some approximations must be made by gridding the set $\mathbf{\Delta}_\rho$ with finite number L of points $\{\rho_i\}_{i=1}^{L}$ (Lee, 1997). Moreover, the infinite variables $\mathbf{P}_\Omega(\rho)$ and $\mathbf{Q}_\Omega(\rho)$ in LMIs (37) and (38) are approximated by combinations of scalar basis functions such as

$$\mathbf{P}_\Omega(\rho) = \sum_{j=1}^{N_P} \phi_j(\rho) P_j \succ 0$$

$$\mathbf{Q}_\Omega(\rho) = \sum_{j=1}^{N_Q} \varphi_j(\rho) Q_j \succ 0$$

where $P_j = P_j^\mathrm{T}$, $Q_j = Q_j^\mathrm{T}$. There is a large freedom in the choice of basis functions (Wood, 1995). For this example, the following choice is made:

$$\{\phi_j\}_{j=1}^{13} = \{\varphi_j\}_{j=1}^{13} = \{1, \rho_1, \rho_2, \rho_1^2, \rho_2^2, \rho_1\rho_2, \rho_1\rho_2^2, \rho_1^2\rho_2,$$
$$\rho_1^2\rho_2^2, \rho_1^3, \rho_2^3, \rho_1\rho_2^3, \rho_1^3\rho_2\}$$

The main consequence of this approximation is that the number of LMIs to be solved is finite and is $2L(2^{s+1} + 1)$ where s is the number of parameters, i.e. $s = 2$.

The full-order controller is designed using the procedure developed in Scherer et al. (1997). Then, an eighth-order controller K_{full} is obtained. The proposed method is used to reduce K_{full}. Then, K_{FWBT} is produced.

To test its effectiveness, a comparison with the method developed in Widowati et al. (2004) (LPV BT), namely K_{BT}, is performed. The feasibility problems (37) and (38) are convex. Using Matlab LMI Control Toolbox, controllers are reduced to the fifth-order obtained heuristically by trial-and-error approach. Therefore, a frequency and time analysis are performed.

5. Results and discussion

The first evaluation is represented in Table 2 where first the assumptions (45) and (46) are checked. Note that for 25 frozen values of (ρ_1, ρ_2), the rank assumption is satisfied for all these points. This fact confirms Remark 3.1 and allows us compute an upper bound. Precisely, Table 3 expresses the upper bounds and $\mathscr{H}_\infty$-norm of the error

Table 3. Parameter values.

Parameter		Value	
Sprung mass	(m_s)	470	[kg]
Unsprung mass	(m_{us})	110	[kg]
Tyre stiffness coefficient	(k_t)	86378	[N/m]
Spring stiffness coefficient	(k_s)	270000	[N/m]

Table 2. Values of the $\|T_{zw}(\rho) - \hat{T}_{zw}(\rho)\|_{i,2}$ and the upper bound for several frozen values of (ρ_1, ρ_2).

ρ_1/ρ_2	−1.0	−0.5	0.0	0.5	1.0
0.0	$1.36 \leq 5.07$	$1.35 \leq 3.32$	$1.33 \leq 4.08$	$1.31 \leq 3.37$	$1.30 \leq 3.72$
0.25	$0.32 \leq 9.88$	$0.32 \leq 9.93$	$0.32 \leq 10.38$	$0.31 \leq 10.39$	$0.31 \leq 9.80$
0.5	$0.00 \leq 1.79$	$0.00 \leq 0.13$	$0.00 \leq 2.22$	$0.00 \leq 11.81$	$0.00 \leq 2.03$
0.75	$0.31 \leq 9.97$	$0.31 \leq 9.92$	$0.30 \leq 10.58$	$0.30 \leq 9.86$	$0.29 \leq 9.89$
1.5	$1.22 \leq 3.77$	$1.20 \leq 3.04$	$1.19 \leq 4.46$	$1.18 \leq 2.88$	$1.17 \leq 2.90$

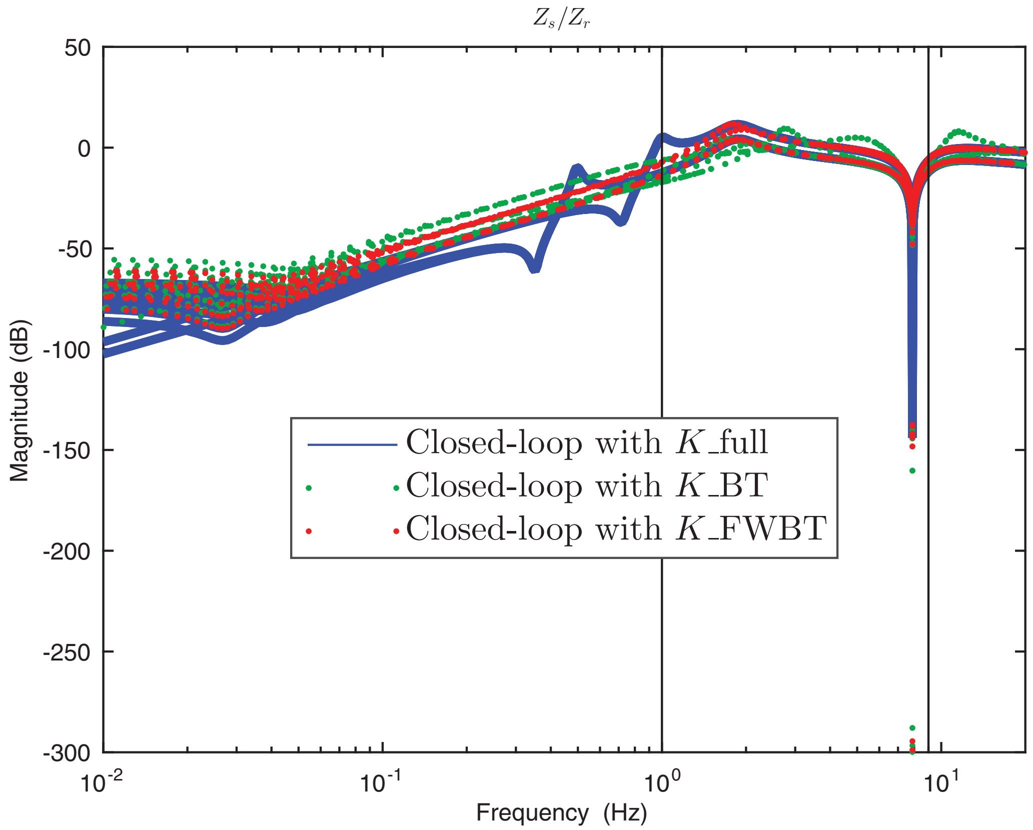

Figure 5. Bode diagram magnitude of the transfer $T_{z_s z_r}$.

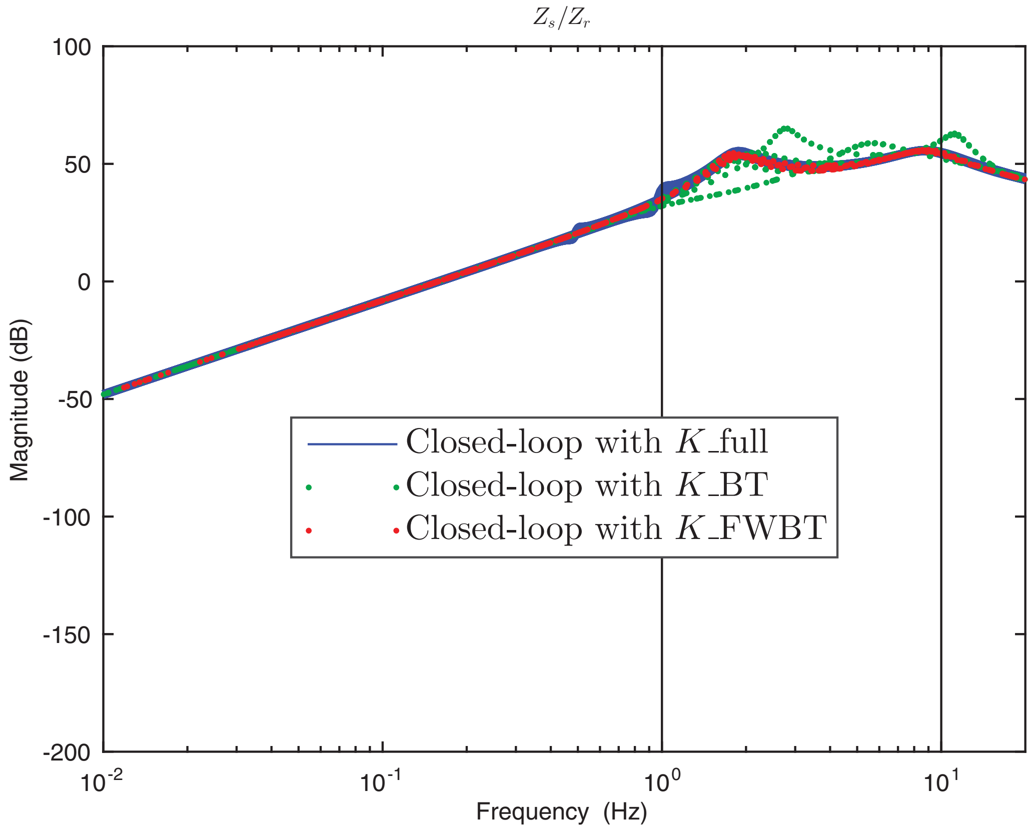

Figure 6. Bode diagram magnitude of the transfer $T_{\ddot{z}_s z_r}$.

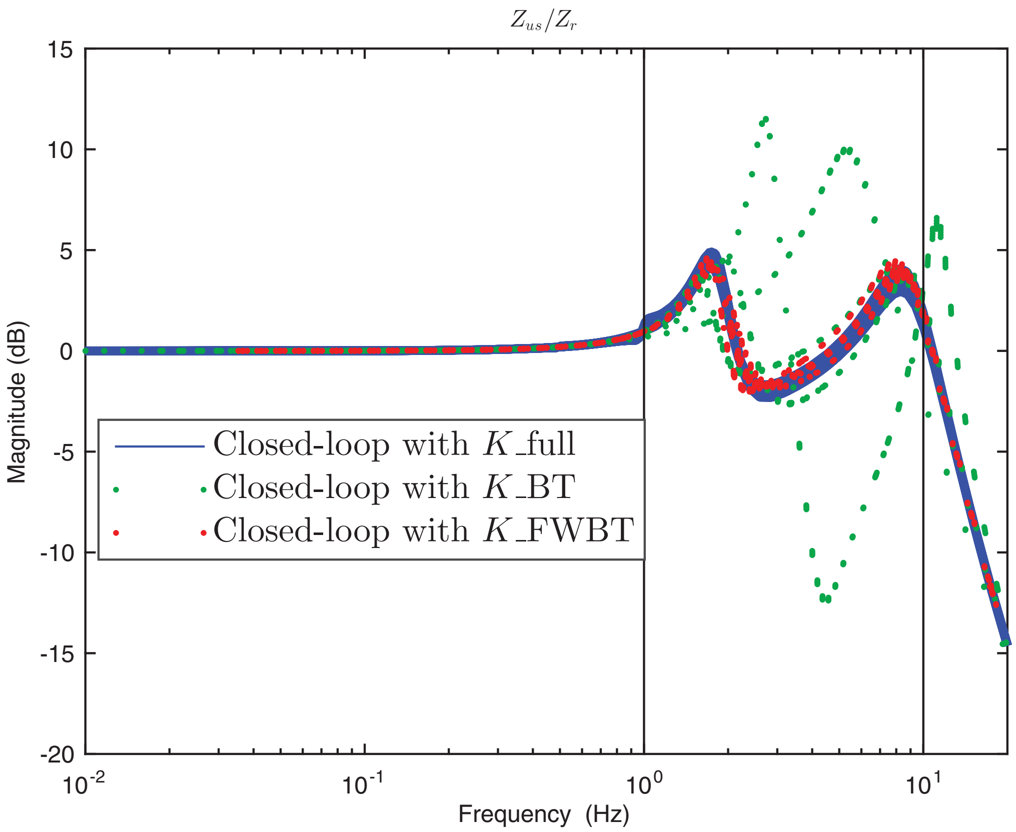

Figure 7. Bode diagram magnitude of the transfer $T_{z_{us}z_r}$.

values. These results evince that the upper bound is correctly positioned (the gap is positive). However, we note that this upper bound is not tight to error H_∞-norm.

5.1 Frequency analysis

The Bode diagrams at several frozen values of ρ_1 and ρ_2 (25 points) of the three transfer functions T_{z_s}, $T_{\ddot{z}_s}$ and $T_{z_{us}}$ are shown, respectively, in Figures 5–7. In fact, the frequency behaviours of the chassis position z_s and acceleration $\ddot{z}_r$ are chosen to be analysed in order to observe the comfort performance regarding the road profile input z_r. The wheel position signal z_{us} is also analysed to test the road holding. Then, the weighting functions $W_{\ddot{z}_s}$ and W_{z_s} designed in Section 4.2 limit the amplification of the previously cited transfers in low-frequency range (around [1 ; 10] Hz). In fact, the human sensitivity to vertical vibrations is important in this frequency range (Do et al., 2010). For this reason, the frequency interval of the proposed frequency-limited FWBT method is chosen as [1 ; 8] Hz.

In Figure 5, note the reduced-order closed-loop system produced by FWBT approximates well the full-order closed-loop system in the chosen frequency range [1 ; 10] Hz. In this interval, the reduced-order closed-loop system produced by the unweighted BT fails. In fact, an important gap appears as 2 and 3 Hz which is exacted as the BT is known to guarantee good approximation at high frequency. The same comment is given in Figure 6 where FWBT fits the full-order closed-loop system in all the shown range unlike BT method that misses the peak around 2 (a resonance frequency). These results are more explicit when observing Figure 7. Indeed, FWBT gives a good approximation when BT fails (2 and 8 Hz). The other important fact stated by these results is the sensitivity against the parameters variation. Actually, a dispersion of ρ_1 and ρ_2 values is induced by the reduction step. This degradation is expected since the given application is a qLPV system where ρ_1 and ρ_2 are depending on the state vector. Then, every decrease in the order (states) affects the ρ_1 and ρ_2 values. However, this loss when reducing is under control for two reasons: the first one, the stability of reduced-order closed loop is preserved and the error is guaranteed limited. The second one is that this dispersion is weak in the required frequency range. Thus, performance is not affected.

Remark 5.1: As Bode diagram for nonlinear systems is not possible, a pseudo-Bode plot is then proposed in Poussot-Vassal (2008)

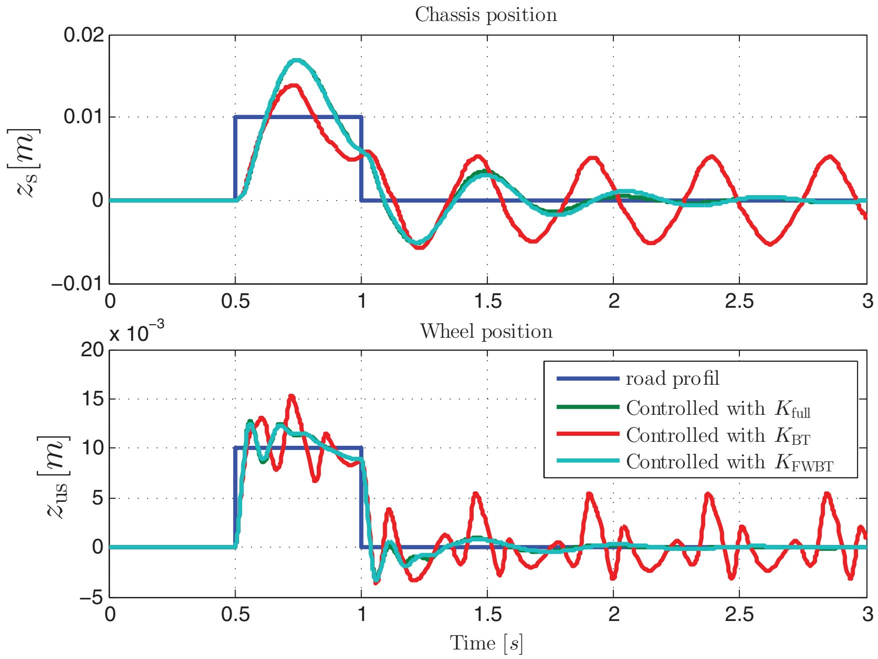

Figure 8. Time response on a bump of 0.01 m x2 m.

5.2 Time analysis

In the time domain, the several controlled suspensions are travelling a bump of 0.01 m x2 m for a vehicle speed 8.3 m/s (i.e. 30 km/h).

It is observed that the time response confirms the contribution of the FWBT reduction method. In fact, in Figure 8, the chassis is stabilised rapidly (1 s after the perturbation) without overtake on the suspension unlike the suspension with K_{BT}. This observation preserves the required 'comfort' performance. Moreover, for z_{us}, the wheel equipped with a K_{FWBT} suspension keeps almost the same profile of the road although its variations which which respects the 'road-handling' performance. The K_{BT} suspension generates an infinite perturbation just after the bump (after 1 s). Note also that temporal test draws two output signals (chassis and wheel positions) regarding the input (the road profile) and by the way there will be just one plot of each transfer besides the several plots in the frequency responses.

6. Conclusion

In this paper, the problem of LPV controller order reduction is investigated, and the approximation in a limited frequency range is considered. based on a similar work on the LTI case, a new frequency-limited method for LPV controller order reduction is derived. The obtained reduced-order controller is proven stable and the degradation in the closed-loop performance is guaranteed bounded. An application of the proposed method to reduce the order of a semi-active suspension controller is performed. The obtained results show its effectiveness to preserve the closed-loop stability and to respect the required performance specifications. The comparison with another existent LPV order reduction technique based on the the classical BT, confirms the contribution of the proposed method.

Acknowledgments

The author would like to address their acknowledgement to the associate editor and the anonymous reviewers for their valuable comments that have helped significantly improve this paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

Anderson, B., & Liu, Y. (1989). Controller reduction: concepts and approaches. *IEEE Transactions on Automatic Control, 34*(8), 802–812.

Anderson, B.D., & Clements, D.J. (1981). Algebraic characterization of fixed modes in decentralized control. *Automatica, 17*(5), 703–712.

Ankelhed, D., Helmersson, A., & Hansson, A. (2011). A partially augmented Lagrangian method for low order H-infinity controller synthesis using rational constraints. In IEEE Conference on Decision and Control and European Control Conference, CDC-ECE'11 (pp. 8219–8224), Orlando, Florida: IEEE.

Apkarian, P., & Gahinet, P. (1995). A convex characterization of gain-scheduled H_∞ controllers. *IEEE Transactions on Automatic Control, 40*(5), 853–864.

Apkarian, P., Gahinet, P., & Becker, G. (1994). Self-scheduled H_∞ linear parameter-varying systems. In *IEEE Transactions on Automatic Control*, American Control Conference (pp. 856–860), Maryland, United States: IEEE.

Briat, C. (2015). *Linear parameter-varying and time-delay systems*, Berlin Heidelberg: Springer.

Do, A., Sename, O., Dugard, L., & Boussaad, S. (2011). Multi-objective optimization by genetic algorithms in H_∞/LPV control of semi-active suspension. In *Proceedings of the 18th IFAC World Congress* (pp. 7162–7167), Milano, Italy: Elsevier.

Do, A., Spelta, C., Savaresi, S., Sename, O., Dugard, L., & Delvecchio, D. (2010). An LPV control approach for comfort and suspension travel improvements of semi-active suspension systems. In Conference on Decision and Control (CDC) (pp. 5560–5565) Atlanta, Georgia: IEEE.

El-Zobaidi, H., & Jaimoukha, I. (1998). Robust control and model and controller reduction of linear parameter varying systems. In Proceedings of the 37th IEEE Conference on Decision and Control, (Vol. 3, pp. 3015–3020).

Enns, D. (1984). Model reduction with balanced realizations: An error bound and a frequency weighted generalization . In The 23rd IEEE Conference on Decision and Control, 1984. (Vol. 23, pp. 127–132), Las Vegas, NV.

Gawronski, W., & Juang, J. (1990). Model reduction in limited time and frequency intervals. *International Journal of Systems Science, 21*(2), 349–376.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their l_∞ error bounds. *International Journal of Control, 39*(6), 1115–1193.

Goddard, P.J. (1995). *Performance-preserving controller approximation* (PhD thesis). Trinity College, Cambridge.

Gugercin, S., & Antoulas, A. (2004). A survey of model reduction by balanced truncation and some new results. *International Journal of Control, 77*, 748–766.

Guo, S., Yang, S., & Pan, C. (2006). Dynamic modeling of magneto-rheological damper behaviors. *Journal of Intelligent Material Systems and Structures, 17*(1), 3–14.

Imran, M., Ghafoor, A., & Sreeram, V. (2014). A frequency weighted model order reduction technique and error bounds. *Automatica, 50*(12), 3304–3309.

Kim, S., Anderson, B., & Madievski, A. (1995). Error bound for transfer function order reduction using frequency weighted balanced truncation. *Systems & Control Letters, 24*, 183–192.

Kong, L. (2012). Controller reduction for linear systems (PhD thesis). Louisiana State University, Baton Rouge.

Lee, L.H. (1997). Identification and robust control of linear parameter-varying systems (PhD thesis). University of California at Berkeley, California.

Li, X., Yin, S., & Gao, H. (2014). Passivity-preserving model reduction with finite frequency H_∞ approximation performance. *Automatica, 50*(9), 2294–2303.

Li, X., Yu, C., & Gao, H. (2015). Frequency-limited $H - infinity$ model reduction for positive systems. *IEEE Transactions on Automatic Control, 60*(4), 1093–1098.

Lin, C., & Chiu, T. (1990, May). Model reduction via frequency weighted balanced realization. In *American Control Conference* (pp. 2069–2070), San Diego, California.

Liu, Y., & Anderson, B. (1989). Singular perturbation approximation of balanced systems. In *Proceedings of the 28th IEEE Conference on Decision and Control, 1989* (Vol. 2, pp. 1355–1360), Tampa, FL: IEEE.

Moore, B. (1981). Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Transactions on Automatic Control, 26*(1), 17–32.

Moore, J.B., Telford, A.J., & Ly, U.L. (1988). Controller reduction methods maintaining performance and robustness. In *Conference on Decision and Control.*

Mullis, C., & Roberts, R. (1976). Synthesis of minimum round-off noise fixed point digital filters. *IEEE Transactions on Circuits and Systems, 23*(9), 551–562.

Packard, A., & Kantner, M. (1996). Gain scheduling the LPV way. In *Proceedings of the 35th IEEE Conference on Decision and Control, 1996* (Vol. 4, pp. 3938–3941), Kobe, Japan: IEEE.

Pernebo, L., & Silverman, L. (1982). Model reduction via balanced state space representations. *IEEE Transactions on Automatic Control, 27*, 382–387.

Vassal, C. (2008). Robust LPV multivariable automotive global chassis control (Theses). Institut National Polytechnique de Grenoble – INPG, Grenoble.

Sandberg, H., & Rantzer, A. (2004). Balanced truncation of linear time-varying systems. *IEEE Transactions on Automatic Control, 49*(2), 217–229.

Scherer, C., Gahinet, P., & Chilali, M. (1997). Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control, 42*(7), 896–911.

Sename, O., Gaspar, P., & Bokor, J. (2013). *Robust control and linear parameter varying approaches: Application to vehicle dynamics*, Berlin Heidelberg. Springer.

Shaker, H.R., & Wisniewski, R. (2011). Generalized Gramian framework for model/controller order reduction of switched systems. *International Journal of Systems Science*, 42, 1277–1291.

Shamma, J., & Athans, M. (1990). Analysis of gain scheduled control for nonlinear plants. *IEEE Transactions on Automatic Control, 35*(8), 898–907.

Sreeram, V., & Anderson, B. (1995, December). Frequency weighted balanced reduction technique: A generalization and an error bound. In *Proceedings of the 34th IEEE Conference on Decision and Control, 1995.* (Vol. 4, pp. 3576–3581).

Tudon-Martınez, J.C., Fergani, S., Varrier, S., Sename, O., Dugard, L., Morales-Menendez, R., & Ramirez-Mendoza, R. (2013). Road adaptive semi-active suspension in a pick-up truck using an LPV controller. In *7th IFAC International Symposium on Advances in Automotive Control (AAC)*, Tokyo.

Varga, A., & Anderson, B. (2001). Accuracy enhancing methods for the frequency-weighted balancing related model reduction. In *Proceedings of the 40th IEEE Conference on Decision and Control, 2001* (Vol. 4, pp. 3659–3664), Orlando, FL: IEEE.

Wang, G., Sreeram, V., & Liu, W. (1999). A new frequency-weighted balanced truncation method and an error bound. *IEEE Transactions on Automatic Control, 44*(9), 1734–1737.

Widowati, W., Riyanto, B., & Bambang, T. (2006). Model Reduction of LPV control with bounded parameter variation rates. In *6th Asian Control Conference*, (pp. 289–296), Bali, Indonesia: IEEE.

Widowati, W., Riyanto, B., & Saragih, R. (2004). Controller reduction of parameter dependent systems. *ITB Journal of Engineering Science, 36*, 43–56.

Wood, G.D. (1995). Control of parameter-dependent mechanical systems (PhD thesis), University of Cambridge,. St Johns College.

Wood, G., Goddard, P., & Glover, K. (1996). Approximation of linear parameter-varying systems. In *Proceedings of the 35th IEEE Conference on Decision and Control, 1996.* (Vol. 1, pp. 406–411).

Zhou, K. (1995, June). A comparative study of H_∞ controller reduction methods. In *Proceedings of the American Control Conference, 1995*, (Vol. 6, pp. 4015–4019), Seattle, Washington: IEEE

Zhou, K., D'Souza, C., & Cloutier, J.R. (1995). Structurally balanced controller order reduction with guaranteed closed loop performance. *Systems & Control Letters, 24*(4), 235–242.